Lévy Flights in External Force Fields: Langevin and Fractional Fokker–Planck Equations, and their Solutions

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We consider Lévy flights subject to external force fields. This anomalous transport process is described by two approaches, a Langevin equation with Lévy noise and the corresponding generalized Fokker–Planck equation containing a fractional derivative in space. The cases of free flights, constant force and linear Hookean force are analyzed in detail, and we corroborate our findings with results from numerical simulations. We discuss the non–Gibbsian character of the stationary solution for the case of the Hookean force, i.e. the deviation from Boltzmann equilibrium for long times. The possible connection to Tsallis's q–statistics is studied.

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I. INTRODUCTION

In recent years there has been growing interest in anomalous diffusion in various fields of physics and related sciences. In one dimension anomalous diffusion is characterized by a mean square displacement of the form

$$\langle (\Delta x)^2 \rangle \propto 2Dt^\gamma, \tag{1}$$

deviating from the linear dependence on time found for Brownian motion [4–6]. The generalized diffusion constant has the dimension $[D] = \text{cm}^2 \text{sec}^{-\gamma}$.

Subdiffusive transport ($0 < \gamma < 1$) is encountered in a diversity of systems, including the charge carrier transport in amorphous semiconductors [7,8], N.M.R. diffusometry on percolation structures [9], and the motion of a bead in a polymer network [10]. On fractal structures in general, subdiffusion prevails due to the occurrence of holes of all length scales [5]. Examples of enhanced diffusion ($\gamma > 1$) include tracer particles in vortex arrays in a rotating flow [11], layered velocity fields [12], and Richardson diffusion [13].

Lévy flights are used to model a variety of processes, such as bulk mediated surface diffusion [14] and applications in porous glasses and eye lenses [15], transport in micelle systems or heterogeneous rocks [16], special problems in reaction dynamics [17], Single Molecule Spectroscopy [18], and even the flight of an albatross [19].

Among the different frameworks for describing anomalous diffusion are fractional Brownian motion [20], the continuous time random walk scheme [7,21], fractional diffusion equations [22,23], generalized Langevin and Fokker–Planck equations (FPEs) [24–28], and generalized thermostatistics [29,30]. Common to all these approaches is the violation of the Central Limit Theorem of probability theory [31,32], and this is achieved either by correlations or by long–tailed statistics. Lévy statistics [31,32] is an example of the latter, and has been used extensively to model both enhanced and dispersive diffusion [6,21,33]. The two most fundamental properties of the Lévy distributions are the stability under addition, following from the Generalized Central Limit Theorem valid for Lévy distributions, and the asymptotic power law decay. These features are responsible for the anomalous character of the diffusion processes we have in mind.

A fractional Fokker–Planck equation (FFPE) describing anomalous transport close to thermal equilibrium, was presented recently [28]. Since it describes subdiffusion in the force–free case, it involves a strong, i.e. slowly decaying memory. In the present paper, we focus on FFPEs which are connected with Lévy flights, and are based on the following Langevin equation for the coordinate $x(t)$ [24,25]:

$$\frac{d}{dt} x(t) = \frac{F(x)}{\gamma m} + \eta(t). \tag{2}$$

Here, m is the mass of the diffusing particle, and γ denotes a friction coefficient. $F(x)$ is the external force field. For simplicity we shall work in one dimension, with obvious modifications in the general case. The noise $\eta(t)$ is the source of the anomalous behavior. We assume $\eta(t)$ to be uncorrelated at different times, and to obey Lévy statistics [34]. In Fourier space we thus define

$$p(k) = \int d\eta e^{-ik\eta} p(\eta) = \exp(-D|k|^{\mu}), \tag{3}$$

where $0 < \mu < 2$. The probability density function (PDF) $p(\eta)$ has an asymptotic power–law behavior according to $p(\eta) \sim |\eta|^{-1-\mu}$ [31,32,35]. Here and in the following, we denote the Fourier transform of a function by using the explicit dependence on the wave number k, and analogously u for the Laplace transform of a t–dependent function. For the special case $\mu = 2$ in Eq. (3), i.e. for a Gaussian noise, we are led back to the Brownian case. In this case $2D$ is the variance of the PDF, but even in the general case the parameter D characterizes the width of the PDF in some sense.

The Langevin equation Eq. (2) is a stochastic differential equation. Often it is more convenient to work with the deterministic equation for the distribution function, the FPE [36,37]. For the power–law noise $\eta(t)$ defined through Eq. (3), we are led to the following FFPE [24,25]:

$$\frac{\partial}{\partial t} W(x,t) = -\frac{\partial}{\partial x} \left(\frac{F(x)W(x,t)}{\gamma m} \right) + D\nabla^{\mu} W(x,t). \tag{4}$$

Here, D denotes the generalized diffusion coefficient with the dimension $[D] = \mathrm{cm}^{\mu}\mathrm{sec}^{-1}$. The Riesz fractional derivative in Eq. (4) is defined through its Fourier–transform [38,39]

$$\nabla^{\mu} = -\int \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} |k|^{\mu} \tag{5}$$

in d dimensions. Note that in the FFPE Eq. (4) the first order differential operator acting upon the force term is not affected by the introduction of the Lévy distribution Eq. (3), see Ref. [40] where a unifying derivation of FFPEs from a generalized master equation is discussed.

In the following we will consider the FFPE Eq. (4) for the three cases of the free flight ($F = 0$), the constant force $F(x) = F_0$, and the Hookean force $F(x) = -\lambda x$, comparing to the Brownian case as we go along. We will discuss the differences to the subdiffusive FFPE of Ref. [28], where a fractional operator in time is encountered and the spatial part of the standard FPE remains unchanged, as well as the possible connection to Tsallis's q–statistics. Numerical simulations corroborate our theoretical findings. We then exemplify the method of solution for the Langevin equation Eq. (2) for a linear force with an additional drift term. Before drawing the conclusions, we give some remarks on the simulations. Some additional calculations on the nature of the correlation functions are presented in the Appendix.

II. FREE LÉVY FLIGHT

In this case we have to solve the anomalous diffusion equation

$$\frac{\partial}{\partial t} W(x,t) = D\nabla^{\mu} W(x,t). \tag{6}$$

Fourier transforming Eq. (6) and utilizing the definition of the fractional Riesz operator Eq. (5) we have

$$\frac{\partial}{\partial t} W(k,t) = -D|k|^{\mu} W(k,t) \tag{7}$$

with the solution

$$W(k,t) = e^{-Dt|k|^{\mu}}, \tag{8}$$

demanding the sharp initial condition $x(0) = 0$, corresponding to $W(x,0) = \delta(x)$ or $W(k,0) = 1$. Comparing to Eq. (3) we recognize the characteristic function of the Lévy distribution, and we thus find in real space the stable law L_{μ}:

$$\begin{aligned} W(x,t) &= (Dt)^{-1/\mu} L_{\mu} \left(\frac{|x|}{[Dt]^{1/\mu}} \right) \\ &= \frac{\pi}{\mu|x|} H_{2,2}^{1,1} \left[\frac{|x|}{(Dt)^{1/\mu}} \left| \begin{array}{l} (1,1/\mu),(1,1/2) \\ (1,1),(1,1/2) \end{array} \right. \right]. \end{aligned} \tag{9}$$

In Eq. (9), we have expressed the Lévy distribution exactly in terms of Fox' H–functions [41,42]. This result Eq. (9) is expected, due to the stable law nature of the underlying Lévy distribution. The asymptotic behavior of the propagator $W(x,t)$ can be derived from Eq. (9) and reads

$$W(x,t) \sim \frac{Dt}{|x|^{1+\mu}} \tag{10}$$

for $|x|^{\mu}/[Dt] \gg 1$, and thus we encounter a divergence of the mean square displacement at all times: $\langle x^2(t)\rangle = \infty$. This is intuitively clear due to the occurrence of arbitrarily long jumps in the Lévy flight, see Fig. 1. Mathematically, the divergence is evident from Eq. (8) by using the properties of the characteristic function: $\langle x^n(t)\rangle = i^n d^n W(k,t)/dk^n|_{k=0}$.

In order to extract the scaling form implied by Eq. (9) operationally, one could enclose the "walker" in an imaginary growing box (see Sec. VI):

$$\langle x^2(t)\rangle_L \sim \int_{L_1 t^{1/\mu}}^{L_2 t^{1/\mu}} dx\, x^2 W(x,t) \sim t^{2/\mu} . \tag{11}$$

This has been implemented numerically, and as can be seen from Fig. 2, where we have a straight line on a log–log plot of $\langle x^2\rangle_L$ as a function of t, for a fixed μ, the expected power law index $2/\mu$ according to Eq. (11) is found. This scaling result $\langle x^2(t)\rangle_L \sim t^{2/\mu}$ is not to be confused with the mean square displacement $\langle x^2\rangle = \int dx\, x^2 W(x,t) = \infty$ for the Lévy flight, resulting from Eq. (10). However, for $\mu > 1$, the squared absolute mean

$$\langle |x|\rangle^2 = \left(\int dx\, |x| W(x,t) \right)^2 \tag{12}$$

converges, and is proportional to $\langle x^2(t)\rangle_L$ from Eq. (11), see the discussion in Ref. [43]. Fig. 3 shows this proportionality for $\mu = 1.5$ [44]. In Fig. 4, we graph the slope $2/\mu$ as a function of μ for a variety of values for the Lévy index μ, and we obtain excellent agreement with Eq. (11). In the case $\mu = 2$ we see from Eq. (8) and the results of the simulations in Fig. 4 that the usual Brownian behavior is recovered. Especially, we obtain the mean square displacement

$$\langle x^2(t)\rangle = 2Dt. \tag{13}$$

The properties of free Lévy flights could also be obtained directly from Eq. (2) employing the method of characteristic functions. This view also allows us to extract the distribution of speeds, which turns out to be Lévy distributed. From this likewise follows the mean kinetic energy, and we have for a finite mass m of the walker in the case $\mu < 2$

$$\langle \frac{1}{2}mv^2\rangle = \infty. \tag{14}$$

III. CONSTANT FORCE: DRIFT AND ACCELERATION

For a constant force $F(x) = F_0$, the FFPE Eq. (4) reads

$$\frac{\partial}{\partial t}W(x,t) = -\frac{\partial}{\partial x}\left(\frac{F_0 W(x,t)}{\gamma m} \right) + D\nabla^{\mu}W(x,t). \tag{15}$$

Returning to the Fourier domain, we recover the equation

$$\frac{\partial}{\partial t}W(k,t) = \left(-ik\frac{F_0}{\gamma m} - D|k|^{\mu} \right) W(k,t), \tag{16}$$

which for the propagator, i.e. $W(k,t)$ with the initial condition $W(k,0) = 1$, yields

$$W(k,t) = \exp\left(-t\left[ik\frac{F_0}{\gamma m} + D|k|^{\mu} \right] \right). \tag{17}$$

This is the same Lévy distribution as calculated for the free Lévy flight in Eq. (8), but at the translated coordinate

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$$W(x,t) = W_0 \left(x - \frac{F_0 t}{\gamma m}, t \right). \tag{18}$$

Here W_0 refers to the distribution of the free Lévy flight. The displacement of the coordinate is due to the balancing of the friction against the imposed constant force, i.e. $\gamma m v = F_0$, in the Galilei transformed system $x \to x - F_0 t/[\gamma m]$. Clearly, the analytical form for the solution in x–space is still given by Eq. (9), but now for the translated coordinate. Thus Lévy flights in a constant force field described by Eq. (15) are similar to (anomalous) diffusion in a constant velocity field [45,46]. The reason why we go directly to the steady state described by $\gamma m v = F_0$ is due to the omission of the inertial term in the Langevin equation Eq. (2). By including this term it can be shown that we obtain a transient contribution. Thus the diffusion in a force field is only equivalent to diffusion in a constant velocity field for large times, according to the discussion in Ref. [45], see below. In fact the same situation is encountered even for the standard diffusion–advection picture [45,46,48]. For long waiting periods in the random walk picture for the subdiffusive model, the external force can only act upon the walker, when it is released from a trap, which leads to the sublinear dependence $\langle x(t) \rangle \propto t^\gamma$ found in Ref. [45], where γ denotes the power law index of the broad waiting time distribution. The Poissonian waiting time distribution, on the other hand, which is typical for the Lévy flyer, leads to the same behavior Eq. (21) below [for $1 < \mu \leq 2$], as known from the Brownian case, due to the very short waiting periods in between jumps. The long–tailed nature of the Lévy flight however makes all higher moments infinite. Both effects lead to an accelerated time dependence of the motion. Including the inertial term in the Langevin equation Eq. (2) changes the solution for short times according to

$$\begin{aligned} x(t) &= \frac{F_0}{m\gamma} \left(t - \frac{1 - e^{-\gamma t}}{\gamma} \right) \\ &\quad + \int_0^t ds\, \eta(s) \left(1 - e^{-\gamma(t-s)} \right). \end{aligned} \tag{19}$$

At times much greater than the characteristic time γ^{-1}, we have

$$x(t) \simeq \frac{F_0}{m\gamma} t + \int_0^t ds\, \eta(s), \tag{20}$$

which follows from the behavior of the Laplace transform of the convolution integral in the second summand of Eq. (19): $\frac{\eta(u)}{u} \left(1 - \frac{u}{u+\gamma} \right) \sim \frac{\eta(u)}{u}$ in the limit $u \ll \gamma$. Thus the effect of the inertial term is negligible for times t much greater than γ^{-1}.

If the first moment exists, that is for $1 < \mu \leq 2$, we find from Eq. (17)

$$\langle x(t) \rangle = \frac{F_0 t}{\gamma m}. \tag{21}$$

Only for $\mu = 2$ we have a finite second moment, and the mean square displacement becomes $\langle [(x(t) - \langle x(t) \rangle)]^2 \rangle = 2Dt$, in agreement with Eqs. (13) and (18).

For the standard FPE as well as for the subdiffusive FFPE introduced in [28], one finds the generalized Einstein relation [28,47,48] $\langle x(t) \rangle_{F_0} = F_0 \langle x^2(t) \rangle_0/(2k_B T)$, relating the first moment in presence of the force F_0 to the second moment in absence of the force. For the Lévy flight model defined in Eq. (15), only in the Brownian limit $\mu = 2$ this relation is satisfied, provided we choose the proper amplitude of the noise, i.e. take it to be thermal noise: $D = k_B T/[\gamma m]$. Generally since we have a diverging mean square displacement, the generalized Einstein relation does not hold, and we have a violation of the classical fluctuation–dissipation theorem.

IV. LINEAR FORCE AND NON–GIBBSIAN STATIONARY SOLUTION

In the case of ordinary Brownian motion the diffusing particles can be trapped in a harmonic potential and thus attain an equilibrium distribution with a finite variance [48,36]. More precisely this equilibrium distribution is the Gibbs or Boltzmann distribution, also obtainable from maximizing the Gibbs entropy under the constraints of norm and energy conservation. Also for subdiffusive transport in a harmonic potential, this property is fulfilled [28]. For Lévy flights we shall see that a stationary solution does exist, however it possesses *no* finite variance. This deviation from the Gibbs–Boltzmann equilibrium implies that Lévy flights do not describe systems close to thermal equilibrium.

For the Hookean force $F(x) = -\lambda x$, corresponding to the harmonic potential $V(x) = \frac{1}{2}\lambda x^2$, the FFPE Eq. (4) becomes

$$\frac{\partial}{\partial t}W(x,t) = \frac{\partial}{\partial x}\left(\frac{\lambda}{\gamma m}xW(x,t)\right) + D\nabla^{\mu}W(x,t). \tag{22}$$

In Fourier–space, the conjugate equation reads

$$\frac{\partial}{\partial t}W(k,t) = -\frac{\lambda}{\gamma m}k\frac{\partial}{\partial k}W(k,t) - D|k|^{\mu}W(k,t), \tag{23}$$

which can be easily solved by making a transformation of variables (applying the method of characteristics)

$$W(k,t) = \exp\left(-\frac{\gamma m D|k|^{\mu}}{\mu\lambda}\left[1 - e^{-\mu\lambda t/\gamma m}\right]\right). \tag{24}$$

This is still a Lévy distribution, only with a different "width" $D \to \frac{\gamma m D}{\mu\lambda}(1 - e^{-\mu\lambda t/\gamma m})$, and the exact solution in real space can again be obtained from Eq. (9) by inserting the time–dependent width. For $\mu = 2$ we recover the Brownian results, but in the general case $\mu < 2$ a different situation arises. We always reach a stationary distribution

$$W_{\mathrm{st}}(k) = \exp\left(-\frac{\gamma m D|k|^{\mu}}{\mu\lambda}\right), \tag{25}$$

but with a diverging mean square. The exact stationary solution in x–space can be given in terms of Fox's H–functions:

$$W_{\mathrm{st}}(x) = \frac{\pi}{|x|}H_{2,2}^{1,1}\left[\frac{|x|^{\mu}\mu\lambda}{D\gamma m}\left|\begin{array}{l}(1,1),\left(1,\frac{\mu}{2}\right)\\(1,\mu),\left(1,\frac{\mu}{2}\right)\end{array}\right.\right], \tag{26}$$

leading to the asymptotic power–law behavior $W_{\mathrm{st}}(x) \sim D\gamma m/(\mu\lambda|x|^{1+\mu})$.

A numerical result for a simulation of a Lévy flight in an harmonic potential is shown in Fig. 5. The slope is in good agreement with the theoretical prediction.

We pause to mention that we could have derived the solution Eq. (24) also by means of a separation ansatz, i.e. assuming a particular solution of the form $W_n(x,t) = T_n(t)\varphi_n(x)$, as was discussed in Refs. [28,45]. Thus we arrive at the ordinary differential equations

$$\frac{d}{dt}T(t) = -\lambda_n T(t) \tag{27a}$$

$$\lambda_n\varphi_n(x) + \frac{d}{dx}\left[\frac{\lambda}{\gamma m}x\varphi_n(x)\right] + D\nabla^{\mu}\varphi_n(x) = 0 \tag{27b}$$

with the eigenvalue λ_n. The complete solution is then given by the sum $W(x,t) = \sum_{n=0}^{\infty}W_n(x,t)$. For the time behavior we find the usual exponentially decaying modes, $T_n(t) = e^{-\lambda_n t}$, and for the spatial eigenfunction we have

$$\varphi_n(k) = c_n|k|^{\lambda_n\gamma m/\lambda}e^{-|k|^{\mu}/\mu}. \tag{28}$$

The eigenvalues are given by $\lambda_n = \frac{\lambda}{\gamma m}\mu n$, and the complete solution in wave number space is given by

$$W(k,t) = e^{-D\gamma m|k|^{\mu}/[\mu\lambda]}$$
$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{D\gamma m}{\mu\lambda}\right)^{n}|k|^{\mu n}e^{-\mu n\lambda t/[\gamma m]}. \tag{29}$$

The sum converges to Eq. (24), and can be transformed back to real space by use of the Fox functions:

$$W(x,t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{D\gamma m}{\mu\lambda}\right)^{n}\frac{\pi}{|x|}\mu^{\lambda n/[\gamma m]}e^{-\mu n\lambda t/[\gamma m]}$$
$$\times H_{2,2}^{1,1}\left[\frac{\mu\lambda}{D\gamma m}|x|^{\mu}\left|\begin{array}{l}\left(1 - \frac{\lambda n}{\gamma m},1\right),\left(1,\frac{\mu}{2}\right)\\(1,\mu),\left(1,\frac{\mu}{2}\right)\end{array}\right.\right]. \tag{30}$$

In comparison to Risken's result for the Fokker–Planck equation in a harmonic potential in the Brownian case [36], also referred to as the Ornstein–Uhlenbeck process, the eigenvalues in the solution Eq. (30) for $\mu = 2$ take on only

even numbers. This is due to our consideration of the start in the origin, so that all the uneven Hermite polynomials occurring in the solution given in Ref. [36], vanish: $H_{2n+1}(0) = 0$. We note that the Fox functions in Eq. (30) can be considered as generalized Hermite polynomials. Clearly, the stationary solution corresponding to $\lambda_0 = 0$ obtained from Eq. (30) is the same stable law as in Eq. (26). Also it can be seen, that only in the Brownian case $\mu = 2$ do we recover the Boltzmann distribution $W(x) \propto e^{-\lambda x^2/[2k_B T]}$. Thus Boltzmann equilibrium with a finite variance is not reached, in spite of the fact that the system is isolated and time–independent in respect to the ensemble. This can be *physically* understood as a consequence of the diverging mean kinetic energy of the free Lévy flight. In this case we have

$$\frac{d}{dt}v(t) = -\gamma v(t) + \gamma \eta(t). \tag{31}$$

If we take the noise to be white according to Eq. (3) with $\mu = 2$, we get from Eq. (31) that in the stationary state

$$\langle v^2 \rangle = D\gamma \Leftrightarrow \langle E_{kin} \rangle = \frac{m\gamma D}{2}. \tag{32}$$

When the external harmonic potential is turned on, a *length scale* is introduced by the comparison $\langle E_{kin} \rangle \approx \langle E_{pot} \rangle = \frac{1}{2}\lambda\langle x^2 \rangle$. This means that $\langle x^2 \rangle \approx mD\gamma/\lambda$. In fact, solving Eq. (2) in the Brownian case with the harmonic potential, we obtain exactly $\langle x^2 \rangle = mD\gamma/\lambda$, in accordance with the equipartition theorem. Similar considerations remain valid for the subdiffusive FFPE from Ref. [45], which thus describes anomalous systems close to thermal equilibrium. However, in the Lévy case we have

$$\langle E_{kin} \rangle = \infty, \tag{33}$$

and therefore the length scale which appears when the potential is introduced, is also diverging. Consequently the question arises whether other statistics could predict the equilibrium distribution in the present context. We here consider the recently proposed Tsallis's q–statistics [29], according to which the generalized entropy

$$S_q[p(x,v)] = \frac{1 - \int p^q(x,v)\,dxdv}{q - 1} \tag{34}$$

is introduced along with the generalized constraints

$$\int p(x,v)\,dx = 1 \tag{35a}$$

$$\int p^q(x,v)E(x,v)\,dx = U; \tag{35b}$$

for $q \to 1$, S_q recovers the usual Boltzmann entropy. Here v is the velocity of the particle and $E(x,v)$ is its energy. Thus Eq. (35b) is a generalized constraint of conservation of energy along with the usual norm conservation Eq. (35a). Varying Eq. (34) subject to these constraints by introducing Lagrange–multipliers, one obtains the stationary distribution

$$p_q(x,v) \sim \left(1 - (1-q)\frac{\beta}{2}\left[\lambda x^2 + mv^2\right]\right)^{1/(1-q)} \tag{36}$$

Integrating over all velocities to obtain the distribution of positions alone yields

$$p_q(x) \sim \left(1 - (1-q)\frac{\beta}{2}\lambda x^2\right)^{\frac{3-q}{2-2q}}, \tag{37}$$

compare to Ref. [30]. Matching this expression to the asymptotic behavior of the stationary solution of the FFPE, Eq. (25), we obtain

$$\mu = \frac{4 - 2q}{q - 1} \tag{38}$$

The relationship Eq. (38) implies that q can range in the interval $(1, 2)$. This is at variance with the relation $\mu = (3 - q_{\text{free}})/(q_{\text{free}} - 1)$ found in the case of the free Lévy flight [30,33], where the allowed range is $q_{\text{free}} \in (5/3, 3)$.

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Note that the q–μ relation does not involve the potential strength λ. Thus, the Tsallis index q makes a jump for a given Lévy index μ, when the potential is switched off, irrespectively of how slowly, e.g. quasi–statically, the limit $\lambda \to 0$ is performed. Moreover, away from the asymptotic regime, the solution predicted by the entropy Eq. (34) does not agree with the solution found in Eq. (25) in the stationary state. Thus we conclude, that the Tsallis entropy is not the appropriate framework for Lévy flights in a harmonic potential described by the generalized Fokker–Planck equation Eq. (4). This form of a generalized entropy does not give rise to the solution of Eq. (25). Recently, it has also been shown that Tsallis q–statistics is by no means unique [49], so that it would have been rather surprising if this very special form of statistics had led to the complete description according to the Lévy flight model. Finally, the FFPE Eq. (4) being linear, is not compatible with the non–extensive nature of Tsallis entropy [29]; compare with the non–linear diffusion equation derived from Tsallis entropy in Ref. [50].

By comparing the distribution of the particle in the harmonic potential $W(x,t)$ with that of the free flight $W_0(x,t)$, we obtain the correspondence

$$W(x,t) = W_0\left(x, t_{\text{eff}}\right), \tag{39}$$

where we define

$$t_{\text{eff}} \equiv \frac{m\gamma}{\mu\lambda}\left(1 - e^{-\mu\lambda t/[\gamma m]}\right). \tag{40}$$

Thus the distribution of the particle position in an harmonic potential can be obtained from the distribution in the free Lévy flight case at an earlier, "effective" time t_{eff}. This comparison illustrates the slowing down of the particle in the harmonic potential, where the restoring force is centered towards the origin. It characterizes in a precise way the approach to stationarity, which is graphed in Fig. 6. Furthermore, this approach is seen to take longer time the smaller μ is; the quickest relaxation occurs in the Brownian case $\mu = 2$.

We conclude this section by some remarks about the inertial term encountered for the Hookean force. There are two time scales (decay times) involved in this case:

$$\tau_s^{-1} \equiv \frac{\gamma}{2}\left(1 - \sqrt{1 - \frac{4\lambda}{m\gamma^2}}\right) \tag{41}$$

$$\tau_f^{-1} \equiv \frac{\gamma}{2}\left(1 + \sqrt{1 - \frac{4\lambda}{m\gamma^2}}\right) \tag{42}$$

Considering only the case of large over damping, i.e. $\gamma^2 \gg 4\lambda/m$, the two time scales separate into a fast and a slow decaying mode, $\tau_f \to \gamma^{-1}$ and $\tau_s \to \gamma m/\lambda$, with $\tau_f \ll \tau_s$. It can be easily shown, that neglecting the fast mode for times long compared to τ_f, corresponds to neglecting the inertial term in the original Langevin equation. Thus we have an effective separation into three different regimes, and being in the second with $\tau_f \ll t < \tau_s$, the approach to the stationary state is not influenced by the omission of the inertial term.

V. SOLUTION OF THE LANGEVIN EQUATION

All of the above results could have been reached equally well directly from the Langevin equation Eq. (2). To illustrate this, we solve this equation for a Lévy flight in a constant force field in addition to a linear force, $F(x) = -\lambda x + F_0$ corresponding for instance to a harmonic potential and a superimposed gravity field. However, it could also correspond to many harmonic oscillators placed at different positions, i.e.

$$F(x) = \sum_{i=1}^{N} -\lambda_i(x - x_i) = -\left(\sum_{i=1}^{N} \lambda_i\right)x + \sum_{i=1}^{N} \lambda_i x_i$$
$$= -\lambda x + F_0. \tag{43}$$

The solution of Eq. (2) is given by

$$x(t) = e^{-\lambda t/[\gamma m]} \int_0^t dt' e^{\lambda t'/[\gamma m]}\left(\eta(t') + \frac{F_0}{\gamma m}\right). \tag{44}$$

The distribution can then be found using the identity

$$p(x,t) = \langle \delta(x - x(t)) \rangle$$
$$= \int \frac{dk}{(2\pi)} \langle \exp(ik[x - x(t)]) \rangle$$
$$= \int \frac{dk}{(2\pi)} e^{ikx} p(k,t). \tag{45}$$

Using the solution Eq. (44) to obtain for the characteristic function $p(k,t)$

$$p(k,t) = \langle \exp \left(-ie^{-\lambda t/[\gamma m]} k \right.$$
$$\left. \times \int_0^t dt' e^{\lambda t'/[\gamma m]} \left[\eta(t') + \frac{F_0}{\gamma m} \right] \right) \rangle, \tag{46a}$$

we have after discretizing the integral

$$p(k,t) \simeq e^{-ikF_0/\lambda(1-e^{-\lambda t/[\gamma m]})}$$
$$\times \langle \prod_{t'=0}^{t} \exp \left(-ie^{-\lambda(t-t')/[\gamma m]} \Delta k \eta(t') \right) \rangle \tag{46b}$$

or

$$p(k,t) \simeq e^{-ikF_0/\lambda(1-e^{-\lambda t/[\gamma m]})}$$
$$\times \prod_{t'=0}^{t} \langle \exp \left(-ie^{-\lambda(t-t')/[\gamma m]} \Delta k \eta(t') \right) \rangle. \tag{46c}$$

Using the definition of Lévy noise from Eq. (3), we obtain

$$p(k,t) \simeq e^{-ikF_0/\lambda(1-e^{-\lambda t/[\gamma m]})}$$
$$\times \prod_{t'=0}^{t} \exp \left(-De^{-\mu\lambda(t-t')/[\gamma m]} \Delta^{\mu} |k|^{\mu} \right) \tag{46d}$$

Reintroducing the integrals and using the same renormalization $D\Delta^{\mu-1} \to D$ as in passing from Eq. (2) to Eq. (4), we finally have

$$p(k,t) = e^{-ikF_0/\lambda(1-e^{-\lambda t/[\gamma m]})}$$
$$\times \exp \left(-D\gamma m \left[1 - e^{-\mu\lambda t/[\gamma m]} \right] \frac{|k|^{\mu}}{\mu\lambda} \right) \tag{47}$$

For $\lambda = 0$, we recover the constant force result, Eq. (17). For $F_0 = 0$ we get Eq. (24). The free Lévy flight result $W_0(x,t)$ according to Eq. (8) is likewise reproduced, when $\lambda = F_0 = 0$. In fact, by comparing to Eq. (8) we have the correspondence

$$W_{\lambda,F_0}(x,t) =$$
$$W_0 \left(x - \frac{F_0}{\lambda} \left[1 - e^{-\lambda t/[\gamma m]} \right], \frac{\gamma m}{\mu\lambda} \left[1 - e^{-\lambda t/[\gamma m]} \right] \right). \tag{48}$$

In the presence of the harmonic potential we can no longer simply make a Galilean transformation in order to eliminate the constant force, since the presence of the linear force singles out another special reference frame.

VI. SOME REMARKS ON THE NUMERICAL SIMULATIONS

Using a computer code written in C, we have simulated Lévy flights in two dimensions in order to compare with the theoretical predictions. The noise has been defined by the asymptotics of the Lévy distribution to

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$$p(\eta) = \frac{\mu\eta_0^{\mu}}{2}|\eta|^{-1-\mu} \tag{49}$$

We have basically investigated three properties: 1) histograms of the distribution of position for the free walker, 2) histograms of the walker in a harmonic potential, and 3) the dynamic exponent as defined via the imaginary box in Eq. (11).

The imaginary box according to Eq. (11) grows in time like the characteristic width of the stable distribution, $\langle x^2(t)\rangle_L \sim t^{2/\mu}$, which is *not* the variance. It gives a measure, that a finite portion of the probability is gathered within a given interval, which we call the imaginary box. The values of L_1 and L_2 have been chosen so as to 1) ensure that we are in the asymptotic regime, where $W(x,t) \sim t|x|^{-1-\mu}$, and 2) to produce good statistics for all values of μ. When fitting the results to a straight line on a log–log plot, we have selected a subset of equidistant points from the entire set of data, in order not to favor the high t region over the low t region.

Concerning the histogram, several precautions have to be taken when working with power law statistics. First of all, due to the occurrence of arbitrarily long steps, we have to define the interval of sampling beforehand, since this is the only way to improve the statistics when increasing the number of samples. We have chosen a minimum limit for the evaluated data points to ensure the asymptotic range. The maximum limit has been chosen as the maximum of the first say 100 (out of a total of 10000) simulations. In this way we obtain many data points throughout the entire region, where the asymptotic power–law expression is valid.

We close with a remark on the axes of the plots of the histograms. To obtain equidistant points on the log–log plots, we chose to graph the histograms of the distributions $p(y)$ of y, where $y = \log x$. If now $p(x) \sim |x|^{-1-\mu}$, we have $p(y) \sim e^{-\mu y}$. Plotting the logarithm of $p(y)$ as a function of y, we find a straight line with slope $-\mu$, which is not to be confused with the slope $-1-\mu$ for the log–log plot of $p(x)$.

VII. CONCLUSIONS

We have investigated Lévy flights under the influence of external force fields. Especially for the cases of free flights, a constant and a linear force explicit solutions are derived and the consequences shown. The solutions can be reduced to a transformation of variables in the free flight result. We have employed the approaches of FFPEs and a Langevin equation with a power law noise term. We have shown that the classical fluctuation–dissipation theorem is violated in the case of constant force, and exhibited the non–Gibbsian nature of the "equilibrium" distribution in the harmonic potential. The approach to stationarity has been characterized by an effective time, and the connection to Tsallis's q–statistics has been explored in some detail. For the constant force, due to the divergence of the mean square displacement, the generalized Einstein relation breaks down, so that the FFPE Eq. (4) also violates linear response. Numerical simulations have been found to agree well with the theoretical analysis.

The direct solution of the Langevin equation Eq. (2) was given, in agreement with the solutions for the FFPE approach. The Langevin approach contains *a priori* more information than the corresponding FFPE, and might be more suitable for a physical interpretation of the underlying system. For a given problem, however, it will be more convenient to employ the FFPE approach and use the methods of characteristics or the separation of variables. Especially, the extraction of moments $\langle x^n\rangle$ is straightforward using the FFPE, by noting that $\frac{d}{dt}\langle x^n\rangle = \int dx\, \dot{W}(x,t)x^n$.

Lévy flights are typical for systems off thermal equilibrium, where thermal equilibrium is to be understood in the classical Boltzmann–Gibbs sense. For the systems analyzed in Refs. [16–19], or similar situations, the remoteness from thermal equilibrium is evident. For the bulk mediated surface diffusion process in Ref. [14,15], one should keep in mind, that the Lévy process emanates only as an *effective* motion, resulting from a Brownian walker which is eventually absorbed on the surface, before he gets activated again.

The connections to the Tsallis's q–entropy still remain unclear. The discrepancy of the relations of q to the Lévy index μ in the free and harmonic cases show, that the concept of q–entropy cannot provide a full explanation of Lévy flights. Especially, the stationary solution found for the harmonic potential, is not an equilibrium solution in the sense of Tsallis's q–entropy. To our understanding, only a non–linear generalized FPE can lead to results compatible with this theory.

We believe that our analysis provides further understanding of anomalous diffusion processes, and will give rise to further experimental investigations, for example Lévy type reaction dynamics subject to an electric field, or the tracer diffusion in rock structures under gravitation, or similar.

VIII. ACKNOWLEDGMENTS

APPENDIX A: CORRELATION FUNCTIONS

Here we examine two–point correlations like $\langle x(t)x(t')\rangle$ in the harmonic case, according to the tools developed in Sec. V. They are often sufficient for the description of the system in applications of the underlying (F)FPE. Assuming the initial condition $x(0) = 0$, we introduce the Green's function

$$x(t) = \int dt' \, G(t,t')\eta(t'), \tag{A1}$$

and by comparison with Eq. (44) it is seen that

$$G(t,t') = e^{-\lambda(t-t')/\gamma m}\Theta(t-t')\Theta(t'). \tag{A2}$$

Due to the divergence of the moments of the Lévy distribution in general, we have to work with characteristic functions discussing correlations:

$$\langle e^{-i\int ds\,A(s)x(s)}\rangle$$
$$= \langle\exp\left(-i\int ds\int ds'\,A(s)G(s,s')\eta(s')\right)\rangle$$
$$\simeq \prod_{s'}\langle e^{-i\Delta\eta(s')\int ds\,A(s)G(s,s')}\rangle$$
$$= \prod_{s'}e^{-D\left|\Delta\int ds\,A(s)G(s,s')\right|^\mu} \tag{A3}$$

where $A(t)$ denotes an a priori arbitrary function, and we used the definition Eq. (3). With the usual renormalization of the noise $D\Delta^{\mu-1} \to D$ [24] we have

$$\langle e^{-i\int ds\,A(s)x(s)}\rangle$$
$$= \exp\left(-D\int ds'\left|\int ds\,A(s)G(s,s')\right|^\mu\right)$$
$$= \exp\left(-D\int ds'\right.$$
$$\times \left.\left|\int ds\int ds''G(s,s')G(s'',s')A(s)A(s'')\right|^{\mu/2}\right)$$
$$= \exp\left(-D\int_0^\infty e^{\mu\lambda s'/[\gamma m]}ds'\left|\int_{s'}^\infty ds\right.\right.$$
$$\times \left.\left.\int_{s'}^\infty ds''e^{-\lambda(s+s'')/[\gamma m]}A(s)A(s'')\right|^{\mu/2}\right). \tag{A4}$$

In the last step we used the identity

$$G(s,s')G(s'',s') = e^{-\lambda(s+s''-2s')/[\gamma m]}$$
$$\times\Theta(s-s')\Theta(s''-s')\Theta(s'). \tag{A5}$$

Putting $A(s) = A\delta(t-s)$, we find

$$\langle e^{-iAx(t)} \rangle$$

$$= \exp\left(-D \int_0^\infty ds' e^{\mu\lambda s'/[\gamma m]} \left| \int_{s'} ds \right.\right.$$

$$\left.\left. \times \int_{s'} ds'' e^{-\lambda(s+s'')/[\gamma m]} A^2 \delta(s-t)\delta(s''-t) \right|^{\mu/2}\right)$$

$$= \exp\left(-D \int_0^\infty ds'\right.$$

$$\left. \times e^{\mu\lambda s'/[\gamma m]} \left| e^{-2\lambda t/[\gamma m]} A^2 \theta(t-s') \right|^{\mu/2}\right)$$

$$= \exp\left(-D \int_0^t ds\, e^{-\mu\lambda(t-s)/[\gamma m]} A^\mu\right)$$

$$= \exp\left(-DA^\mu \left[\frac{\gamma m(1-e^{-\mu\lambda t/[\gamma m]})}{\mu\lambda}\right]\right), \tag{A6}$$

which is equivalent to Eq. (24), as it should be, and therefore includes also the results for the Brownian case. One–point correlation functions (moments of the distribution), if they exist, can thus be obtained from Eq. (A6), so we will proceed to the more interesting case of the two–point correlations. To this end, we take $A(t) = A[\delta(t-t_1) - \delta(t-t_2)]$ and insert it into Eq. (A4), and we find

$$\langle \exp\left(-iA(x(t_1) - x(t_2))\right)\rangle =$$

$$\exp\left(-DA^\mu \left[\frac{\gamma m(1-e^{-\mu\lambda t_2/[\gamma m]})}{\mu\lambda}\right.\right.$$

$$\times \left(1-e^{-\mu\lambda(t_1-t_2)/[\gamma m]}\right)^\mu$$

$$\left.\left.+\frac{\gamma m(1-e^{-\mu\lambda(t_1-t_2)\gamma m})}{\mu\lambda}\right]\right) \tag{A7}$$

for $t_1 > t_2$, and t_1 and t_2 interchanged in the other case when $t_2 > t_1$. This is essentially the characteristic function of the stochastic variable $x(t_1) - x(t_2)$, so all two–point correlation functions and the distribution itself can in principle be found from Eq. (A7). However, it is seen that it is a Lévy distribution, so all higher moments diverge. Nevertheless, Eq. (A7) still gives some information about the correlation between the position of the walker at two different times. When $t_1 \gg t_2$ the characteristic function splits up into the product of the characteristic function of the two variables Eq. (A6) which means, that $x(t_1)$ and $x(t_2)$ are independent in this limit. At intermediate times, i.e. when both t_1 and t_2 are small, the correlation depends on both. This is a memory of the initial conditions, since both walkers start out at the origin. At long times, initial conditions are not important, and hence only a dependence on the time difference $t_1 - t_2$ (kept fixed and finite) is retained,

$$\langle e^{-iA(x(t_1)-x(t_2))} \rangle$$

$$= \exp\left(\frac{-D\gamma m}{\mu\lambda}\left[\left(1-e^{-\mu\lambda|t_1-t_2|/[\gamma m]}\right)^\mu\right.\right.$$

$$\left.\left.+1-e^{-\mu\lambda|t_1-t_2|/[\gamma m]}\right]\right). \tag{A8}$$

Writing $x_{12} \equiv x(t_1) - x(t_2)$, we have by the usual arguments a Lévy distribution of x_{12}:

$$\langle e^{-ik\cdot x_{12}} \rangle = e^{-\tilde{D}_\mu(t_1,t_2)|k|^\mu} \tag{A9}$$

with

$$\tilde{D}_\mu(t_1,t_2) \equiv D\frac{\gamma m}{\mu\lambda}\left[1-e^{-\mu\lambda t_2/[\gamma m]}\left(1-e^{-\mu\lambda(t_1-t_2)/[\gamma m]}\right)^\mu\right.$$

$$\left.+1-e^{-\mu\lambda(t_1-t_2)/[\gamma m]}\right]. \tag{A10}$$

As with the other Lévy distributions, this stochastic variable is characterized by a power law tail ($t_1 \equiv t$, $t_2 \equiv 0$):

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$$p(x_{12}, t) \sim D\frac{\gamma m}{\mu\lambda} \left((1 - e^{-\mu\lambda t/[\gamma m]})^{\mu} \right.$$
$$\left. + 1 - e^{\mu\lambda t/[\gamma m]} \right) |x_{12}|^{-1-\mu}. \tag{A11}$$

In the case of $\mu = 2$ all the preceding results reproduce the well–known Brownian relations for ordinary diffusion. One could proceed like this finding three–point correlations in an analogous manner.

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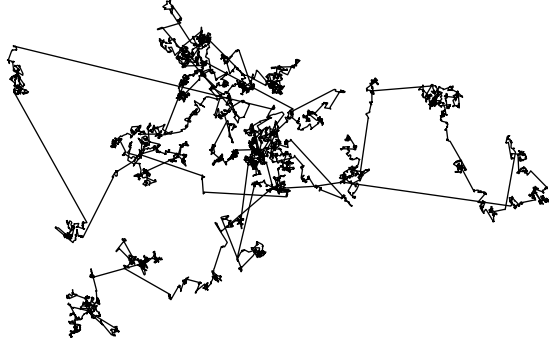[50] A. Compte and D. Jou, J. Phys. A **29**, 4321 (1996)

FIG. 1. Typical Lévy–flight for the Lévy index $\mu = 1.4$. The clustering is obvious. Each cluster is statistically self–similar to the unmagnified picture. The fractal dimension of the flight is $d_f = \mu$ [6,24].
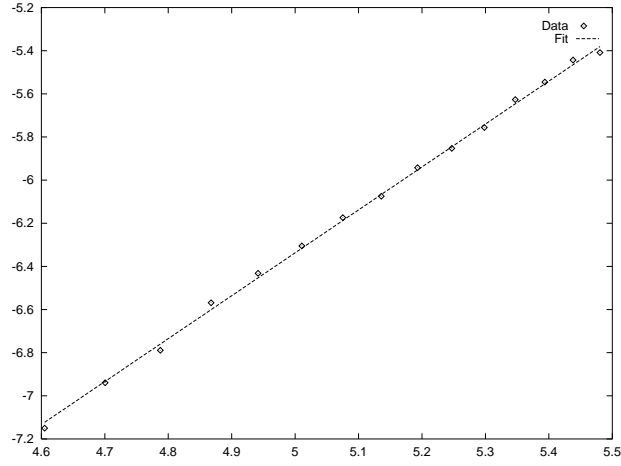

FIG. 2. Function $\langle x^2(t) \rangle_L$ from Eq. (11) versus time with $\mu = 1$ in a log–log plot. The slope of the straight line is 1.990 ± 0.028, which is to be compared to the expected value $2/\mu = 2$.
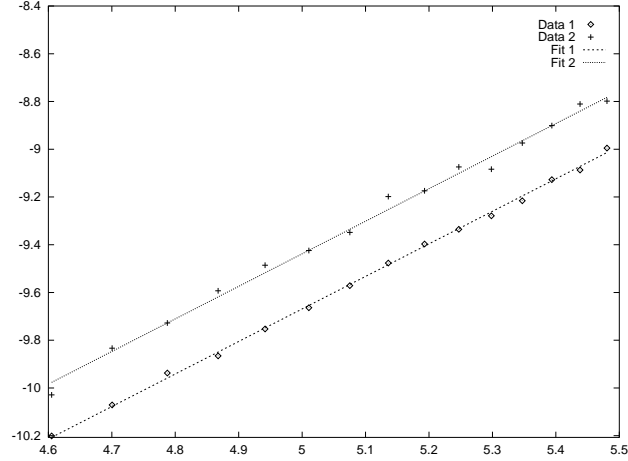

FIG. 3. log–log plot of $\langle x^2 \rangle_L$ versus time from Eq. (11) (lower curve), compared to the squared absolute mean $\langle |x| \rangle^2$ from Eq. (12) (upper curve) for $\mu = 1.5$. The fitted slope is 1.363 ± 0.013 and 1.364 ± 0.035, respectively, which is in good agreement with the theoretical value $4/3$.
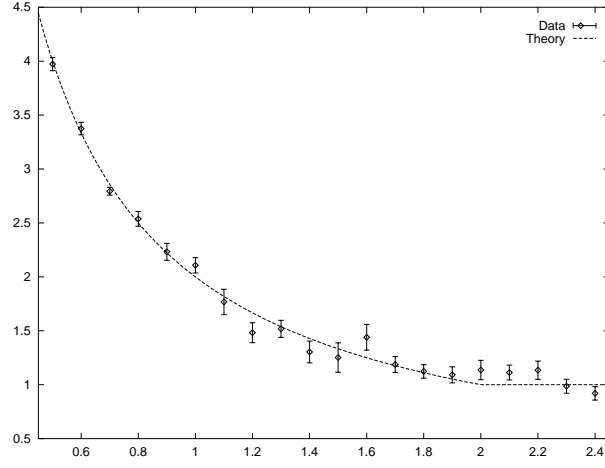
FIG. 4. Graph of the slope $2/\mu$ according to Eq. (11) as a function of the Lévy index μ. Note the bend at $\mu = 2$ marking the transition to normal diffusion.
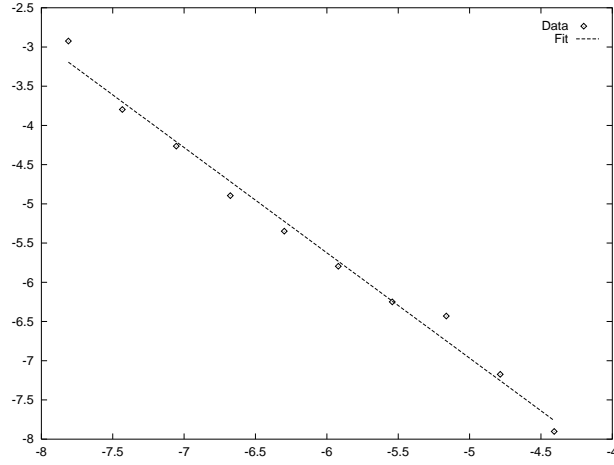

FIG. 5. Histogram for the stationary solution W_{st} of a Lévy flight in a harmonic potential versus $|x|$, as a plot of $\log W_{\mathrm{st}}(y)$ versus y, where $y = \log(x)$ denotes the natural logarithm of the position of the flyer, see Sec. VI. The data were produced for a Lévy index $\mu = 1.4$. The fit indicated by the dashed line reveals a slope of -1.408 ± 0.108, which thus shows a good agreement with the theoretical prediction $-\mu$.
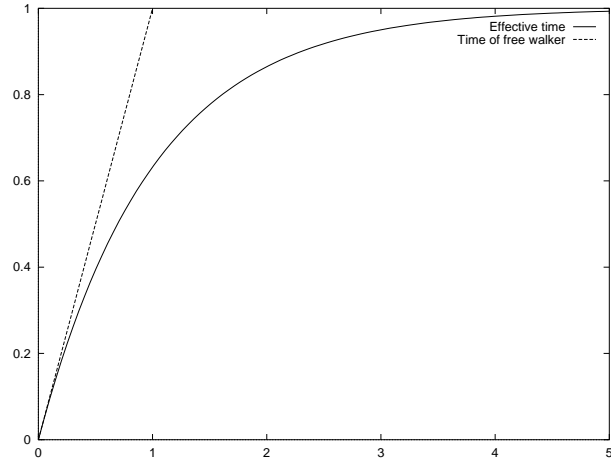

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FIG. 6. The linear time as seen by the free walker (dashed line), compared to the effective time sensed by the random walker in the harmonic potential, as a function of the laboratory time. For the Lévy flight in the potential, the restoring Hookean force slows down the spreading of the diffusion, and eventually brings it to a halt.